

Yang-Mills fields in flux compactifications on homogeneous manifolds with $SU(4)$ -structure

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Abstract

The $Spin(7)$ -instanton equations are natural BPS equations for instantons on 8-manifolds. We study these equations on nearly Kähler and Calabi-Yau torsion manifolds of the form $M \times G/H$, with G/H a coset space and M a product of a torus with Euclidean space. By imposing G -invariance the instanton equations reduce to interesting equations on M ; for example, equations used by Kapustin and Witten in the geometric Langlands program arise in this way. We carry out reductions in a number of examples, and where possible present simple solutions.

1 Introduction

Generalisations of the four-dimensional anti-self-dual Yang-Mills equations to dimensions greater than four were first written down by Corrigan et. al. [1]. The mathematical study of these equations was spurred by the article [2] of Donaldson and Thomas, which sought high-dimensional analogues of low-dimensional gauge-theoretical topological invariants. In physics, these equations turned out to be naturally BPS equations, and have in particular become important in compactifications of string, M-, and F-theory. They have also been used to construct topological field theories [3], whose dimensional reductions have found application both in the geometric Langlands program [4] and an analytic continuation of Chern-Simons theory [5].

The most interesting dimensions seem to be 6, 7, and 8, and the corresponding generalised anti-self-dual equations are respectively called the Hermitian-Yang-Mills or Donaldson-Uhlenbeck-Yau equations [6], the G_2 -instanton equations [7], and the Spin(7)-instanton equations [8]. The first solutions were written down in Euclidean space [9, 10], but later it was realised that the equations make sense on any manifold, provided it is equipped with a G -structure for a suitable Lie group G . One requires $G = \text{SU}(3)$ in dimension 6, $G = G_2$ in dimension 7, and $G = \text{Spin}(7)$ in dimension 8. The choice $G = \text{SU}(4)$ is also suitable on an 8-dimensional almost-complex manifold, because an $\text{SU}(4)$ -structure lifts to a $\text{Spin}(7)$ -structure; the Spin(7)-instanton equations are sometimes called the $\text{SU}(4)$ -instanton equations in this context. For a recent discussion and more references, see for example [11, 12, 13].

Most work on the generalised anti-self-dual Yang-Mills equations has restricted attention to integrable G -structures, that is, Riemannian manifolds with holonomy group G . However, if one is interested in string compactifications with fluxes [14] one should consider instead non-integrable G -structures. The torsion of the G -structure, which measures the failure to be integrable, is in this context identified with a 3-form field in supergravity.

With this in mind, we have undertaken to study the Spin(7)-instanton equations on 8-dimensional coset spaces with non-integrable G -structures. We have restricted most of our attention to nearly Kähler and Calabi-Yau torsion coset spaces (both of which admit an almost complex structure). Mathematically, these are quite natural choices: they are obtained by requiring that the torsion 3-form is the real part of a (3,0)-form or a (2,1)-form respectively. Physically, nearly Kähler 6-manifolds are suitable for compactification of both Romans massive supergravity [15] and heterotic supergravity [16] to AdS_4 , while the Calabi-Yau torsion condition is a necessary (though not sufficient) condition in compactifications of heterotic supergravity to 4-dimensional Minkowski space [17, 18].

An outline of the rest of this article is as follows. Section 2 contains an introduction to $\text{SU}(n)$ -structures on $2n$ -manifolds, and includes a digression on torsion classes in the case $n = 3$. In section 3 we introduce the Spin(7)-instanton equations, highlighting their relations with the torsionful Yang-Mills equations and with topological quantum field theories. In section 4 we describe how to construct G -invariant gauge fields on coset spaces G/H . We then discuss in sections 5-8 the Spin(7)-instanton equations on many examples of compact 8-manifolds of the form $T^d \times G/H$. In each example, we first impose G -invariance on the gauge field, reducing the 8-dimensional Spin(7)-instanton equations to equations on the torus T^d . Where possible, these d -dimensional equations are solved. Besides solutions on the torus, solutions on decompactified spaces $\mathbb{R}^{d-p} \times T^p$ are also considered. We summarise our results and make some closing comments in section 9.

2 Manifolds with $SU(n)$ -structure

2.1 Calabi-Yau torsion and nearly Kähler manifolds

A manifold with H -structure is by definition an m -dimensional manifold M whose tangent bundle TM admits a reduction of its structure group from $GL(m, \mathbb{R})$ to H . Manifolds of dimension $2n$ with $SU(n)$ -structure admit a set of canonical objects, consisting of an almost complex structure J , a Riemannian metric g , a real two-form ω and a complex n -form Ω . With respect to J , the forms ω and Ω are of type $(1,1)$ and $(n,0)$, respectively, and there is a compatibility condition, $g(J\cdot, \cdot) = \omega(\cdot, \cdot)$. With respect to the volume form V_g of g , ω and Ω are normalised so that

$$\omega^n = n!V_g \quad \text{and} \quad \Omega \wedge \bar{\Omega} = (2i)^n V_g . \quad (2.1)$$

Let e^a , where $a = 1, \dots, 2n$, be a local frame for the cotangent bundle, and let $\Theta^\mu = e^{2\mu-1} + ie^{2\mu}$ and $\Theta^{\bar{\mu}} := \bar{\Theta}^\mu$ be local frames for the $(1,0)$ and $(0,1)$ parts of the complexified cotangent bundle. Then J, ω, Ω may be written as follows:

$$J\Theta^\mu = i\Theta^\mu, \quad \omega = \frac{i}{2} \sum_{\mu=1}^n \Theta^\mu \wedge \Theta^{\bar{\mu}}, \quad \Omega = \Theta^1 \wedge \Theta^2 \wedge \dots \wedge \Theta^n . \quad (2.2)$$

One may choose a connection which preserves g, J, ω and Ω . We are particularly interested in the case where the torsion of this connection is totally anti-symmetric, so that the Cartan structure equations may be written

$$d\Theta^\mu = \Gamma_\nu^\mu \wedge \Theta^\nu + \Theta^\mu \lrcorner T . \quad (2.3)$$

Here Γ encodes the Christoffel symbols of the connection: it is a traceless anti-Hermitian matrix with 1-form entries. The 3-form T is the torsion of the connection, while the symbol \lrcorner denotes contraction, defined in terms of the Hodge star by $u \lrcorner v = *(u \wedge *v)$.

Two particularly interesting cases arise when the torsion T is the real part of either a $(3,0)$ -form or a $(2,1)$ -form. In the former case, the manifold is called nearly Kähler [19, 20, 21]. The latter case is equivalent to a property called Calabi-Yau torsion [22], or CYT, as we now explain.

On a Hermitian manifold, the Kähler-torsion (or Bismut) connection is a unitary connection $\tilde{\Gamma}$ on the cotangent bundle with anti-symmetric torsion given by $\tilde{T} = Jd\omega$ (where J acts on the 3-forms according to the rule $J(u \wedge v \wedge w) = (Ju \wedge Jv \wedge Jw)$). A Hermitian manifold is called CYT when the Kähler-torsion (KT) connection has structure group $SU(n)$ rather than $U(n)$ (so that $\tilde{\Gamma}$ is traceless as well as anti-Hermitian).

Now suppose that the torsion T of an $SU(n)$ -structure on some $2n$ -manifold is the real part of a $(2,1)$ -form. The right hand side of (2.3) contains no part of type $(0,2)$. This immediately tells us that the almost complex structure J is integrable, and the manifold is Hermitian. Writing

$$T = \frac{1}{2} T_{\mu\nu\bar{\rho}} \Theta^\mu \wedge \Theta^\nu \wedge \Theta^{\bar{\rho}} + \frac{1}{2} T_{\mu\bar{\nu}\bar{\rho}} \Theta^\mu \wedge \Theta^{\bar{\nu}} \wedge \Theta^{\bar{\rho}}, \quad (2.4)$$

and using the Cartan structure equation (2.3) and the anti-Hermiticity of Γ , one can show that

$$d\omega = \frac{i}{2} ((\Gamma_\nu^\mu \wedge \Theta^\nu + \Theta^\mu \lrcorner T) \wedge \Theta^{\bar{\mu}} - \Theta^\mu \wedge (\Gamma_{\bar{\nu}}^{\bar{\mu}} \wedge \Theta^{\bar{\nu}} + \Theta^{\bar{\mu}} \lrcorner T)) \quad (2.5)$$

$$= -\frac{i}{2} T_{\mu\nu\bar{\rho}} \Theta^\mu \wedge \Theta^\nu \wedge \Theta^{\bar{\rho}} + \frac{i}{2} T_{\mu\bar{\nu}\bar{\rho}} \Theta^\mu \wedge \Theta^{\bar{\nu}} \wedge \Theta^{\bar{\rho}} \quad (2.6)$$

$$= -JT . \quad (2.7)$$

Thus the Kähler-torsion connection $\tilde{\Gamma}$ and the connection Γ preserving the $SU(n)$ -structure have the same torsion $T = \tilde{T} = Jd\omega$. So these two connections must agree, and the manifold is CYT.

2.2 Intrinsic torsion classes in dimension six

Now we restrict attention to the case of 6-manifolds with $SU(3)$ -structure. Chiossi and Salamon have provided a classification of $SU(3)$ -structures in terms of intrinsic torsion classes [23]. Here we explain how nearly Kähler manifolds and CYT manifolds fit into this classification. The five torsion classes can be determined by analysing either the torsion tensor or the exterior derivatives of ω and Ω . Here we follow the latter approach: the torsion classes are defined by

$$d\omega = \text{Im}((W_1^+ - iW_1^-)\Omega) + W_3 + W_4 \wedge \omega, \quad (2.8)$$

$$d\Omega = (W_1^+ + iW_1^-)\omega \wedge \omega + (W_2^+ + iW_2^-) \wedge \omega + \Omega \wedge W_5. \quad (2.9)$$

Here W_1^\pm are real functions, W_4, W_5 are real 1-forms, W_2^\pm are the real and imaginary parts of a $(1,1)$ -form with $\omega \lrcorner W_2^\pm = 0$, and W_3 is the real part of a $(2,1)$ -form with $\omega \lrcorner W_3 = 0$.

The torsion classes of a nearly Kähler manifold are already well-known. Since the space of $(3,0)$ -forms is 1-dimensional, one can write $T = \text{Re } \rho \Omega$ for some complex function ρ . Using (2.3) one can calculate

$$d\omega = \frac{i}{2} \left((\Gamma_\beta^\alpha \wedge \Theta^\beta + \Theta^\alpha \lrcorner \text{Re } \rho \Omega) \wedge \Theta^{\bar{\alpha}} - \Theta^\alpha \wedge (\Gamma_{\bar{\beta}}^{\bar{\alpha}} \wedge \Theta^{\bar{\beta}} + \Theta^{\bar{\alpha}} \lrcorner \text{Re } \rho \Omega) \right) \quad (2.10)$$

$$= 3 \text{Im } \rho \Omega, \quad (2.11)$$

$$d\Omega = \frac{1}{2} \epsilon_{\alpha\beta\gamma} (\Gamma_\delta^\alpha \wedge \Theta^\delta + \Theta^\alpha \lrcorner \text{Re } \rho \Omega) \wedge \Theta^\beta \wedge \Theta^\gamma \quad (2.12)$$

$$= 2\bar{\rho} \omega \wedge \omega. \quad (2.13)$$

Comparing with (2.8), (2.9), one sees that

$$W_2^\pm = W_3 = W_4 = W_5 = 0 \quad (2.14)$$

for a nearly Kähler 6-manifold.

The torsion classes for CYT manifolds can be derived similarly (see also [18]). Since the complex structure J is integrable, $d\Omega$ is a $(3,1)$ -form and the torsion classes W_1 and W_2 vanish. Let $\tilde{\Gamma}$ denote the KT connection, with torsion \tilde{T} . Then $\tilde{\Gamma}$ is a Hermitian matrix with 1-form entries and $\tilde{\Gamma}, \tilde{T}$ satisfy the Cartan structure equations (2.3). The manifold will be CYT if and only if $\tilde{\Gamma}$ is traceless. One may calculate $d\Omega$ as follows:

$$d\Omega = \frac{1}{2} \epsilon_{\alpha\beta\gamma} (\tilde{\Gamma}_\delta^\alpha \wedge \Theta^\delta + \Theta^\alpha \lrcorner \tilde{T}) \wedge \Theta^\beta \wedge \Theta^\gamma \quad (2.15)$$

$$= \delta_\alpha^\delta (\tilde{\Gamma}_\delta^\alpha + 2\delta^{\alpha\bar{\alpha}} \tilde{T}_{\bar{\alpha}\bar{\beta}\delta} \Theta^{\bar{\beta}}) \wedge \Omega \quad (2.16)$$

$$= (\tilde{\Gamma}_\alpha^\alpha + \omega \lrcorner d\omega) \wedge \Omega. \quad (2.17)$$

We recall that W_4 and W_5 are determined by $W_4 = \frac{1}{2} \omega \lrcorner d\omega$ and $d\Omega = \Omega \wedge W_5$. So $\tilde{\Gamma}$ is traceless if and only if $2W_4 + W_5 = 0$. Therefore a CYT manifold satisfies

$$W_1 = 0, \quad W_2^\pm = 0, \quad 2W_4 + W_5 = 0. \quad (2.18)$$

The CYT property does not place any restriction on W_3 .

3 The Spin(7)-instanton equations

3.1 Spin(7)-structure and instanton equations

A Spin(7)-structure on an 8-manifold is specified by a self-dual 4-form Σ satisfying certain conditions. Given any SU(4)-structure on an 8-manifold, there is an S^1 's-worth of compatible Spin(7)-structures, determined by

$$\Sigma = \text{Re } e^{i\xi} \Omega + \frac{1}{2} \omega \wedge \omega , \quad (3.1)$$

with ξ a real parameter. Conversely, given such a Spin(7)-structure, the space of compatible SU(4)-structures is $\text{Spin}(7)/\text{SU}(4) = \text{Spin}(7)/\text{Spin}(6) = S^6$.

A natural generalisation of the 4-dimensional anti-self-dual Yang-Mills equation to an 8-dimensional manifold with Spin(7)-structure is the Σ -anti-self-dual equation:

$$*\mathcal{F} = -\Sigma \wedge \mathcal{F} . \quad (3.2)$$

When the Spin(7)-structure is determined by an SU(4)-structure via (3.1), the equations can be written [2, 24]:

$$e^{-i\xi} \mathcal{F}_{\mu\nu} = -\frac{1}{2} \epsilon_{\mu\nu\rho\sigma} \mathcal{F}_{\bar{\rho}\bar{\sigma}} , \quad (3.3)$$

$$\omega \lrcorner \mathcal{F} = 0 . \quad (3.4)$$

Here we have written the field strength tensor \mathcal{F} in a basis of (1,0)- and (0,1)-forms:

$$\mathcal{F} = \frac{1}{2} \mathcal{F}_{\mu\nu} \Theta^\mu \wedge \Theta^\nu + \mathcal{F}_{\bar{\mu}\bar{\nu}} \Theta^\mu \wedge \Theta^{\bar{\nu}} + \frac{1}{2} \mathcal{F}_{\bar{\mu}\bar{\nu}} \Theta^{\bar{\mu}} \wedge \Theta^{\bar{\nu}} . \quad (3.5)$$

Anti-hermiticity of \mathcal{F} implies that $\mathcal{F}_{\bar{\mu}\bar{\nu}} = -\mathcal{F}_{\mu\nu}^\dagger$ and $\mathcal{F}_{\mu\bar{\nu}}^\dagger = \mathcal{F}_{\nu\bar{\mu}}$.

Note that the Spin(7)-instanton equations are weaker than the Hermitian-Yang-Mills equations:

$$\mathcal{F}_{\mu\nu} = 0 , \quad (3.6)$$

$$\omega \lrcorner \mathcal{F} = 0 . \quad (3.7)$$

Any solution of the Hermitian-Yang-Mills equations is automatically a solution of the Spin(7)-instanton equations.

We will sometimes need to be able to determine whether two SU(4)-structures are compatible with the same fixed Spin(7)-structure. The (1,1)-form $\tilde{\omega}$ of a compatible SU(4)-structure should satisfy

$$*\tilde{\omega} = \frac{1}{3} \Sigma \wedge \tilde{\omega} . \quad (3.8)$$

The space of solutions to this equation is 7-dimensional, so the space of normalised solutions is S^6 . Any solution $\tilde{\omega}$ of (3.8) is a linear sum of ω and a solution f of the complex self-dual equation:

$$e^{-i\xi} f_{\mu\nu} = \frac{1}{2} \epsilon_{\mu\nu\rho\sigma} f_{\bar{\rho}\bar{\sigma}} . \quad (3.9)$$

3.2 Yang-Mills action functional

Taking the exterior derivative of the Σ -anti-self-dual equation (3.2) and using the Bianchi identity gives

$$D * \mathcal{F} + \mathcal{F} \wedge d\Sigma = 0 , \quad (3.10)$$

where D is the covariant derivative and $d\Sigma$ is Hodge dual to a torsion 3-form. The torsionful Yang-Mills equation (3.10) is the variational equation for the action,

$$S = - \int \text{Tr} (\mathcal{F} \wedge * \mathcal{F} + \mathcal{F} \wedge \mathcal{F} \wedge \Sigma) . \quad (3.11)$$

The Σ -anti-self-dual equation (3.2) can be derived from this action using a Bogomolny argument: there is a lower bound on the action,

$$S \geq 0 , \quad (3.12)$$

which is saturated if and only if (3.2) holds.

The proof of (3.12) in the integrable case $d\Sigma = 0$ appeared in [3], and is easily adapted to the case $d\Sigma \neq 0$. One can orthogonally decompose the curvature $\mathcal{F} = \mathcal{F}^+ + \mathcal{F}^-$ into two parts satisfying $\Sigma \wedge \mathcal{F}^- = - * \mathcal{F}^-$ and $\Sigma \wedge \mathcal{F}^+ = 3 * \mathcal{F}^+$. The lower bound follows from integrating the identity,

$$-\text{Tr}(\mathcal{F} \wedge * \mathcal{F} + \mathcal{F} \wedge \mathcal{F} \wedge \Sigma) = -4 \text{Tr}(\mathcal{F}^+ \wedge * \mathcal{F}^+) , \quad (3.13)$$

and discarding the positive piece $-\int \text{Tr}(\mathcal{F}^+ \wedge * \mathcal{F}^+)$. Clearly the inequality is satisfied if and only if $\mathcal{F}^+ = 0$.

3.3 Reduction to four dimensions and the geometric Langlands program

Now we will consider some translation-invariant cases of the $\text{Spin}(7)$ -instanton equations on \mathbb{R}^8 .

Let x^μ and y^a be coordinates on \mathbb{R}^n and \mathbb{R}^{8-n} , with $\mu = 1, \dots, n$ and $a = 1, \dots, 8-n$. A gauge field invariant under y -translations is written,

$$\mathcal{A} = A + \Phi = A_\mu(x) dx^\mu + \phi_a(x) dy^a . \quad (3.14)$$

Its curvature is

$$\mathcal{F} = F + D\Phi + \Phi \wedge \Phi , \quad (3.15)$$

with F denoting the curvature of A and $D\Phi = (d\phi_a + [A, \phi_a]) dy^a$ a covariant derivative.

Consider the following $\text{SU}(4)$ -structure in the case $n = 4$:

$$\Theta^\mu = dx^\mu + i dy^\mu . \quad (3.16)$$

The $(2,0)$ -part of the field strength is

$$\mathcal{F}^{2,0} = \frac{1}{8} (F_{\mu\nu} - [\phi_\mu, \phi_\nu] - i(D_\mu \phi_\nu - D_\nu \phi_\mu)) \Theta^\mu \wedge \Theta^\nu . \quad (3.17)$$

If we write $\chi = \phi_\mu dx^\mu$, the complex anti-self-dual equation (3.3) can be cast neatly as

$$(F - \chi \wedge \chi) + tD\chi \quad \text{is anti-self-dual} , \quad (3.18)$$

$$D\chi - t(F - \chi \wedge \chi) \quad \text{is self-dual} , \quad (3.19)$$

where $t = -\tan(\xi/2)$. The second of the Spin(7)-instanton equations (3.4) is

$$D * \chi = 0 . \quad (3.20)$$

The Hermitian-Yang-Mills equations are slightly stronger: they consist of (3.20), and the vanishing of the (2,0)-part (3.17) of \mathcal{F} :

$$F - \chi \wedge \chi = 0 , \quad (3.21)$$

$$D\chi = 0 . \quad (3.22)$$

Equations (3.18)-(3.22), generalised to a curved manifold, were used by Kapustin and Witten in their work on the geometric Langlands program [4]. The Spin(7)-instanton equations (3.18)-(3.20), with complex parameter t , define a topological quantum field theory. When t is real these are just the Spin(7)-instanton equations; when t is non-real they are equivalent, by taking Hermitian and anti-Hermitian parts, to the Hermitian-Yang-Mills equations (3.20)-(3.22). So all of Kapustin and Witten's equations have a natural interpretation as 8-dimensional anti-self-dual equations.

Equations (3.18)-(3.20) have also recently found an application in an analytically continued Chern-Simons theory. We will have more to say about this at the end of this section.

3.4 Reduction to two dimensions and the Hitchin and Nahm equations

Now we consider the case $n = 2$. Let $z = x^1 + ix^2$ be the complex coordinate on $\mathbb{R}^2 = \mathbb{C}$ and let $w^\alpha = y^{2\alpha-1} + iy^{2\alpha}$, $\alpha = 1, 2, 3$, be the complex coordinates on \mathbb{C}^3 . The natural SU(4)-structure (with vanishing torsion) is

$$\Theta^\alpha = dw^\alpha, \quad \alpha = 1, 2, 3, \quad \Theta^4 = dz . \quad (3.23)$$

Throughout this paper we will adopt a convention that indices α, β, γ run from 1 to 3. We again impose translational symmetry via equation (3.14), with

$$\Phi = \phi_\alpha(z, \bar{z})\Theta^\alpha + \phi_{\bar{\alpha}}(z, \bar{z})\Theta^{\bar{\alpha}} . \quad (3.24)$$

So A is a gauge field on \mathbb{C} and ϕ_1, ϕ_2, ϕ_3 are matrix-valued functions, with $\phi_{\bar{\alpha}} = -\phi_\alpha^\dagger$ for anti-Hermiticity.

The Spin(7)-instanton equations (3.3), (3.4) for \mathcal{F} reduce to

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{\alpha}} = \frac{1}{2} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} [\phi_{\beta}, \phi_{\gamma}] , \quad (3.25)$$

$$*F = 2i[\phi_\alpha, \phi_{\bar{\alpha}}] . \quad (3.26)$$

These equations generalise both the Hitchin equations and the Nahm equations. The Hitchin equations are recovered on setting $\phi_2 = \phi_3 = 0$:

$$D_{\bar{z}} \phi_{\bar{1}} = 0 , \quad (3.27)$$

$$*F = 2i[\phi_1, \phi_{\bar{1}}] . \quad (3.28)$$

The Nahm equations arise when ϕ_α are anti-Hermitian: in this case equation (3.26) implies that $F = 0$, so that $A = 0$ in some gauge. Then equation (3.25) (with $\xi = 0$) implies that $\partial_2 \phi_\alpha = 0$ and

$$\partial_1 \phi_\alpha = \epsilon_{\alpha\beta\gamma} [\phi_\beta, \phi_\gamma]. \quad (3.29)$$

3.5 Hermitian flow equations

One further interesting property of equation (3.25) is that it is a “Hermitian flow” on a Hermitian manifold. A Hermitian flow is the analogue for a Hermitian manifold of gradient flow on a Riemannian manifold or Hamiltonian flow on a symplectic manifold. Suppose that we have a Hermitian manifold N with coordinates u^μ and Hermitian metric $h_{\mu\bar{\nu}} du^\mu du^{\bar{\nu}}$. Given a function W on N , one can define a Hermitian flow equation for maps from \mathbb{C} into N :

$$\frac{\partial u^{\bar{\mu}}}{\partial \bar{z}} = h^{\bar{\mu}\nu} \frac{\partial W}{\partial u^\nu}. \quad (3.30)$$

The flow equation can be cast in a coordinate-free fashion as follows:

$$h \left(X, \frac{\partial}{\partial \bar{z}} \right) = \partial W(X) \quad \forall X \in \Gamma(T^{1,0}N). \quad (3.31)$$

Consider the space N of triples of complex matrices (ϕ_1, ϕ_2, ϕ_3) . Holomorphic and anti-holomorphic tangent vectors are written

$$X = X_\alpha \frac{\partial}{\partial \phi_\alpha}, \quad \bar{X} = X_{\bar{\alpha}} \frac{\partial}{\partial \phi_{\bar{\alpha}}}, \quad (3.32)$$

with (X_1, X_2, X_3) again a triple of complex matrices and $X_{\bar{\alpha}} = -X_\alpha^\dagger$. A natural Hermitian metric is

$$h(X, \bar{Y}) = -\text{Tr}(X_\alpha Y_{\bar{\alpha}}). \quad (3.33)$$

Then the Hermitian flow for the real function

$$W(\phi) = -\frac{1}{3} \text{Tr}(\epsilon^{\alpha\beta\gamma} \phi_\alpha \phi_\beta \phi_\gamma + \epsilon^{\bar{\alpha}\bar{\beta}\bar{\gamma}} \phi_{\bar{\alpha}} \phi_{\bar{\beta}} \phi_{\bar{\gamma}}) \quad (3.34)$$

is

$$\frac{\partial \phi_{\bar{\alpha}}}{\partial \bar{z}} = \frac{1}{2} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} [\phi_\beta, \phi_\gamma], \quad (3.35)$$

and this coincides with (3.25) when $A = 0$.

The fact that (3.25) is a Hermitian flow reflects the more general property that the complex anti-self-dual equation (3.3) on $\mathbb{C} \times M^6$ is, for suitable M^6 , a Hermitian flow on the infinite-dimensional space of connections on M^6 . The space N of connections on M^6 inherits a natural complex structure from that on M^6 , and holomorphic and anti-holomorphic vector fields on N are $(1,0)$ - and $(0,1)$ -forms on M^6 , respectively. A natural Hermitian metric is given by

$$h(\psi, \eta) = -4 \int_{M^6} \text{Tr}(\psi \wedge * \eta) \quad \text{for} \quad \psi \in \Lambda^{1,0} M^6, \quad \eta \in \Lambda^{0,1} M^6. \quad (3.36)$$

If $\text{Im}\Omega$ is closed, then the Hermitian flow equation for the function

$$W(\mathcal{A}) = - \int_{M^6} \text{Tr} \left(\mathcal{A} \wedge d\mathcal{A} + \frac{2}{3} \mathcal{A} \wedge \mathcal{A} \wedge \mathcal{A} \right) \wedge \text{Im}\Omega \quad (3.37)$$

is

$$\frac{\partial \mathcal{A}_{\bar{\alpha}}}{\partial \bar{z}} = \frac{1}{2} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} \mathcal{F}_{\beta\gamma} . \quad (3.38)$$

This equation is equivalent to the complex anti-self-dual equation (3.3) (with $\mathcal{A}_{\bar{z}} = 0$). The condition that $\text{Im}\Omega$ is closed implies that the torsion classes W_3, W_5 vanish: thus Calabi-Yau and nearly Kähler manifolds lead to a Hermitian flow, but CYT manifolds only do so if $W_5 = 0$.

We will close this section by making a few more comments on the 4-dimensional Kapustin-Witten equations (3.18)-(3.20). In a recent paper [5] these were found to play a fundamental role in an analytically continued Chern-Simons theory. Witten observed that they emerge as gradient flow equations for the real part $\text{Re}(\mathcal{I})$ of a holomorphic Chern-Simons functional \mathcal{I} for a complex connection on a 3-manifold. The above discussion explains why this should be the case: $\text{Re}(\mathcal{I})$ arises by reducing W from six to three dimensions. The complex connection is $A + i\phi$ in the language of subsection 3.3, and the Hermitian flow in two dimensions reduces to a gradient flow in one dimension.

4 Coset spaces and invariant gauge fields

The standard example of a manifold with H -structure is a coset space G/H . All of the manifolds that we will consider will be products of coset spaces G/H with tori T^d (or decompactified tori $T^{d-p} \times \mathbb{R}^p$), such that the subgroup H of G can be embedded in $\text{SU}(4)$. The natural H -structure can then be lifted to an $\text{SU}(4)$ -structure, which in most of our examples is either nearly Kähler or CYT.

In the first part of this section we will describe the natural H -structure on a coset space G/H , and in the second part we will describe the machinery required to write down G -invariant gauge fields on G/H . Standard references for this material are [25, 26, 27].

4.1 Coset space geometry

Let G/H denote the space of left cosets gH . We choose a frame for the cotangent bundle of G/H as follows. Assume that there exists a subspace \mathfrak{m} of \mathfrak{g} such that $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{m}$ and $[\mathfrak{h}, \mathfrak{m}] \subset \mathfrak{m}$, where $\mathfrak{g}, \mathfrak{h}$ denote the Lie algebras of G and H . Let I_a and I_i be bases for \mathfrak{m} and \mathfrak{h} respectively, where $a = 1, \dots, \dim G/H$ and $i = \dim G/H + 1, \dots, \dim G$. These will be chosen orthonormal with respect to an adjoint-invariant quadratic form q , for example a multiple of the Cartan-Killing form.

The basis elements I_a, I_i determine left-invariant vector fields on G , whose dual left-invariant 1-forms are denoted \hat{e}^a, \hat{e}^i . Over a coordinate patch $U \subset G/H$, we can choose a local section L of the principle bundle $G \rightarrow G/H$ (in other words, a map $L : U \rightarrow G$ such that $\pi \circ L$ is the identity, where $\pi : G \rightarrow G/H$ is the natural projection). The 1-forms \hat{e}^a, \hat{e}^i can be pulled back to 1-forms e^a, e^i on U by the local section.

The 1-forms e^a form a local frame for the cotangent bundle of G/H . We can define a metric on G/H by declaring them to be orthonormal: this is the natural metric induced by the chosen

quadratic form q on \mathfrak{g} . The remaining 1-forms e^i define the so-called canonical connection,

$$A^0 = e^i I_i . \quad (4.1)$$

This is a G -invariant connection on the H -bundle $G \rightarrow G/H$; there is a one-to-one correspondence between G -invariant connections and choices of $\mathfrak{m} \subset \mathfrak{g}$. Since e^a is a frame, one can write $e^i = e_a^i e^a$. The differentials of the 1-forms e^i, e^a are encoded in the Maurer-Cartan equations:

$$de^a = -f_{ib}^a e^i \wedge e^b - \frac{1}{2} f_{bc}^a e^b \wedge e^c , \quad (4.2)$$

$$de^i = -\frac{1}{2} f_{bc}^i e^b \wedge e^c - \frac{1}{2} f_{jk}^i e^j \wedge e^k . \quad (4.3)$$

Here the f 's are the structure constants of the Lie algebra \mathfrak{g} , defined by

$$[I_i, I_j] = f_{ij}^k I_k , \quad [I_i, I_a] = f_{ia}^b I_b , \quad [I_a, I_b] = f_{ab}^i I_i + f_{ab}^c I_c . \quad (4.4)$$

The first Maurer-Cartan equation (4.2) can be interpreted as a Cartan structure equation (2.3), written in real form:

$$de^a = -\Gamma_b^a \wedge e^b + e^a \lrcorner T . \quad (4.5)$$

The connection $\Gamma_b^a = e^i f_{ib}^a$ is just the canonical connection (4.1) acting on 1-forms via the adjoint action of \mathfrak{h} on \mathfrak{m} , so its structure group is H . Its torsion

$$T = -\frac{1}{6} f_{abc} e^a \wedge e^b \wedge e^c \quad (4.6)$$

is a 3-form, because the structure constants $f_{abc} = \delta_{ad} f_{bc}^d$ are totally antisymmetric in a, b, c .

Because the connection Γ has structure group H , there is an underlying H -structure on G/H . Globally this comes about because the cotangent bundle $T^*(G/H)$ is isomorphic to the vector bundle $G \times_H \mathfrak{m}^*$ associated to the H -principle bundle $G \rightarrow G/H$ via the adjoint action of H on the dual \mathfrak{m}^* of \mathfrak{m} .

4.2 G -invariant gauge fields

Bundles over G/H to which the action of G can be lifted are determined by homomorphisms from H into the gauge group K . The images of the generators I_i of H will be denoted L_i , while the whole set of generators of K will be denoted L_A and the structure constants will be denoted h_{AB}^C . Gauge fields invariant under the action of G are determined by linear maps from \mathfrak{m} to the Lie algebra \mathfrak{k} of K , which commute with the action of H . Such maps can be written $I_a \mapsto \Phi_a^A L_A$, and Φ_a^A must satisfy

$$f_{ia}^b \Phi_b^A = h_{iB}^A \Phi_a^B . \quad (4.7)$$

The most general possible G -invariant gauge field on $M \times G/H$ can be written

$$\mathcal{A} = A + A^0 + L_A \Phi_a^A e^a . \quad (4.8)$$

Here A denotes a gauge field on the smooth manifold M of dimension d , with gauge group equal to the centraliser of the image of H in K . The canonical connection is denoted $A^0 = e^i L_i$. We also use Φ as shorthand for $L_A \Phi_a^A e^a$, where Φ_a^A satisfies (4.7) and is allowed to depend on the coordinate on M .

In order to calculate the field strength tensor, we first split the exterior derivative into two parts acting on M and G/H : $d = d^M + d^{G/H}$. Then short calculations using the Maurer-Cartan equations (4.2), (4.3) and the identity (4.7) show that

$$F^0 := d^{G/H} A^0 + A^0 \wedge A^0 = -\frac{1}{2} f_{ab}^i L_i e^a \wedge e^b, \quad (4.9)$$

$$d^{G/H} \Phi + A^0 \wedge \Phi + \Phi \wedge A^0 = \Phi \lrcorner T, \quad (4.10)$$

with T the torsion tensor (4.6). Thus the field strength of \mathcal{A} is given by

$$\mathcal{F} = F + F^0 + D(\Phi_a^A L_A) \wedge e^a + \Phi \wedge \Phi + \Phi \lrcorner T. \quad (4.11)$$

Here we have introduced the covariant derivative $D = d^M + [A, \cdot]$.

The field strength (4.11) should be compared with the translation-invariant field strength on flat space (3.15). In fact (4.11) differs from (3.15) in precisely three ways: by the field strength F^0 of the canonical connection, by the torsion term involving T , and by the constraint (4.7) imposed on Φ . In most of the cases that we consider, F^0 solves the Spin(7)-instanton equations, so that the Spin(7)-instanton equations will differ from the relevant flat cases only by the torsion term and the constraint on Φ .

For the remainder of this section, we will explain why F^0 solves the Spin(7)-instanton equations whenever the subgroup H of G can be embedded in $SU(4)$. Actually, we will show that it solves the stronger Hermitian-Yang-Mills equations. A more explicit proof for the nearly Kähler case can be found in [13].

One way to describe the Hermitian-Yang-Mills equations for a 2-form F in 8 dimensions is to say that F belongs to the Lie sub-algebra $\mathfrak{su}(4)$, when 2-forms are identified with elements of $\mathfrak{so}(8)$ via the metric. We wish to show that the 2-forms $(1/2) f_{ab}^i e^a \wedge e^b$ solve the Hermitian-Yang-Mills equations. These map to matrices $B_i \in \mathfrak{so}(8)$ with entries f_{ia}^b . Then it is obvious that these B_i belong to the sub-algebra $\mathfrak{su}(4)$: the B_i are the generators of $\mathfrak{h} \subset \mathfrak{su}(4)$, acting on vectors via the adjoint action of \mathfrak{h} on \mathfrak{m} .

Note that, if we choose a metric on G/H other than that induced by the Cartan-Killing form, F^0 is no longer guaranteed to solve the Spin(7)-instanton equations. We will see examples of metrics for which F^0 does not solve the Spin(7)-instanton equations in section 8.

5 Yang-Mills fields on parallelisable CYT spaces

5.1 The Calabi-Eckmann complex structure

Our basic examples of CYT manifolds will be products of two odd-dimensional spheres, which may be regarded as a coset spaces:

$$S^{2n-1} \times S^{2m-1} = S(U(n) \times U(m))/S(U(n-1) \times U(m-1)) \quad \text{with } n, m \geq 1. \quad (5.1)$$

These admit natural $S(U(n-1) \times U(m-1))$ -structures, which may be lifted to CYT $SU(n+m-1)$ -structures. There is actually a choice of lift; in our examples we will consider 2-parameter families of CYT structures.

The Calabi-Eckmann complex structure can loosely be described as follows: the spheres S^{2n-1} , S^{2m-1} are regarded as S^1 -fibrations over \mathbb{CP}^{n-1} , \mathbb{CP}^{m-1} . The projective spaces are each equipped

with their standard complex structure, while a suitable complex structure is chosen on the fibre $S^1 \times S^1$. Thus these CYT spaces are elliptic fibrations, so are perhaps suitable compactification spaces for F-theory.

The natural metric to choose on $S^{2n-1} = \mathrm{SU}(n)/\mathrm{SU}(n-1)$ is induced by the Cartan-Killing form on $\mathfrak{su}(n)$, when $n \geq 2$ (S^1 is equipped with the usual round metric). It is worth emphasising that this metric coincides with the usual round metric only in the cases of S^3 and S^1 . We will consider 2-parameter families of product metrics on $S^{2n-1} \times S^{2m-1}$, obtained by rescaling the metrics on each factor. Along with a family of compatible complex structures, these will determine families of CYT structures.

In this section we will consider the Spin(7)-instanton equations on 2 examples of 8-dimensional CYT manifolds whose natural structure group is the trivial group. They are $S^3 \times T^5$ and $S^3 \times S^3 \times T^2$. In the second case there is more than one way to obtain a CYT structure: one may regard it either as a product of two copies of CYT $S^3 \times S^1$, or as a product of $S^3 \times S^3$ with a torus. All gauge and scalar fields will take values in the Lie algebra \mathfrak{k} of the gauge group K , or its complexification $\mathfrak{k}_{\mathbb{C}}$.

5.2 Invariant gauge fields on $(S^3 \times S^3) \times T^2$

The manifold $S^3 \times S^3$ is the quotient of $\mathrm{SU}(2)^2$ by the trivial subgroup $H = \{1\}$, so fits into the discussion of section 4, albeit as a slightly trivial case.

Let $\hat{e}^a, \check{e}^a, a = 1, 2, 3$, be left-invariant 1-forms on two copies of $\mathrm{SU}(2) = S^3$, normalised so that $d\hat{e}^a = -(1/2R_1)\epsilon_{abc}\hat{e}^b \wedge \hat{e}^c$ and $d\check{e}^a = -(1/2R_2)\epsilon_{abc}\check{e}^b \wedge \check{e}^c$, with R_1, R_2 positive real constants which will determine the radii of the two 3-spheres. Let z denote a complex coordinate on T^2 . An $\mathrm{SU}(4)$ -structure on $S^3 \times S^3 \times T^2$ is defined by setting

$$\Theta^1 = \hat{e}^1 + i\hat{e}^2, \quad \Theta^2 = \check{e}^1 + i\check{e}^2, \quad \Theta^3 = \hat{e}^3 + i\check{e}^3, \quad \Theta^4 = dz. \quad (5.2)$$

The $\mathrm{SU}(4)$ -structure is preserved by the canonical connection, whose Christoffel symbols Γ vanish and whose torsion (4.6) is

$$T = -\frac{1}{R_1}\hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 - \frac{1}{R_2}\check{e}^1 \wedge \check{e}^2 \wedge \check{e}^3 \quad (5.3)$$

$$= -\frac{i}{4R_1}\Theta^1 \wedge \Theta^{\bar{1}} \wedge (\Theta^3 + \Theta^{\bar{3}}) + \frac{1}{4R_2}\Theta^2 \wedge \Theta^{\bar{2}} \wedge (\Theta^3 - \Theta^{\bar{3}}). \quad (5.4)$$

Since the torsion is the real part of a (2,1)-form, this $\mathrm{SU}(4)$ -structure is CYT.

We make the usual ansatz (4.8) for an $\mathrm{SU}(2)^2$ -invariant gauge field \mathcal{A} , with $A^0 = 0$ and A a gauge field on T^2 taking values in \mathfrak{k} . We parametrise Φ in terms of scalars ϕ_α , $\alpha = 1, 2, 3$, taking values in $\mathfrak{k}_{\mathbb{C}}$:

$$\Phi = \phi_\alpha \Theta^\alpha + \phi_{\bar{\alpha}} \Theta^{\bar{\alpha}}. \quad (5.5)$$

Here and throughout this article we adopt the notation $\phi_{\bar{\alpha}} = -\phi_\alpha^\dagger$. The field strength tensor \mathcal{F} of \mathcal{A} is given in (4.11) (with T as in (5.3)). The Spin(7)-instanton equations for \mathcal{F} on $S^3 \times S^3 \times T^2$

reduce to vortex-type equations,

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{1}} = [\phi_2, \phi_3] + \frac{1}{2R_2} \phi_2, \quad (5.6)$$

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{2}} = [\phi_3, \phi_1] - \frac{i}{2R_1} \phi_1, \quad (5.7)$$

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{3}} = [\phi_1, \phi_2], \quad (5.8)$$

$$*F = 2i[\phi_\alpha, \phi_{\bar{\alpha}}] + \left(\frac{1}{R_1} + \frac{i}{R_2} \right) \phi_3 + \left(\frac{1}{R_1} - \frac{i}{R_2} \right) \phi_{\bar{3}}, \quad (5.9)$$

on the complex torus T^2 (or one of its decompactifications $S^1 \times \mathbb{R}, \mathbb{R}^2$). These generalise equations (3.25), (3.26). Here $D = d + [A, \cdot]$ denotes the covariant derivative on T^2 . Because H is trivial, equation (4.7) places no restrictions on ϕ_α .

5.3 Invariant gauge fields on $(S^3 \times S^1)^2$

An alternative choice of complex structure on $S^3 \times S^3 \times T^2$ is

$$\Theta^1 = dx^1 + i\check{e}^3, \quad \Theta^2 = dx^2 + i\check{e}^3, \quad \Theta^3 = \hat{e}^1 + i\hat{e}^2, \quad \Theta^4 = \check{e}^1 + i\check{e}^2, \quad (5.10)$$

with x^1, x^2 coordinates on the two copies of S^1 . This $SU(4)$ -structure is also CYT, as can be seen from the canonical torsion:

$$T = -\frac{1}{R_1} \hat{e}^1 \wedge \hat{e}^2 \wedge \hat{e}^3 - \frac{1}{R_2} \check{e}^1 \wedge \check{e}^2 \wedge \check{e}^3 \quad (5.11)$$

$$= -\frac{1}{4R_1} \Theta^3 \wedge \Theta^{\bar{3}} \wedge (\Theta^1 + \Theta^{\bar{1}}) - \frac{1}{4R_2} \Theta^4 \wedge \Theta^{\bar{4}} \wedge (\Theta^2 + \Theta^{\bar{2}}). \quad (5.12)$$

We parametrise the $SU(2)^2$ -invariant gauge field (4.8) as follows:

$$\Phi = \phi_i \Theta^i + \phi_{\bar{i}} \Theta^{\bar{i}} + \chi_1 \hat{e}^3 + \chi_2 \check{e}^3, \quad i = 3, 4. \quad (5.13)$$

Here χ_1, χ_2 take values in \mathfrak{k} , and ϕ_3, ϕ_4 take values in $\mathfrak{k}_{\mathbb{C}}$. Introducing $\chi = \chi_1 dx^1 + \chi_2 dx^2$, the $\text{Spin}(7)$ -instanton equations on $(S^3 \times S^1)^2$ reduce to vortex-type equations,

$$*(F - \chi \wedge \chi - iD\chi) = -4e^{i\xi} [\phi_{\bar{3}}, \phi_{\bar{4}}] \quad (5.14)$$

$$D_1 \phi_3 - \frac{\phi_3}{R_1} - i[\chi_1, \phi_3] = e^{i\xi} \left(D_2 \phi_{\bar{4}} - \frac{\phi_{\bar{4}}}{R_2} + i[\chi_2, \phi_{\bar{4}}] \right) \quad (5.15)$$

$$D_2 \phi_3 - i[\chi_2, \phi_3] = -e^{i\xi} (D_1 \phi_{\bar{4}} + i[\chi_1, \phi_{\bar{4}}]) \quad (5.16)$$

$$*D * \chi - \frac{\chi_1}{R_1} - \frac{\chi_2}{R_2} = 2i[\phi_i, \phi_{\bar{i}}], \quad (5.17)$$

on the torus T^2 . Again, the constraint (4.7) makes no restriction on the fields ϕ_i .

Equations (5.14)-(5.17) are nonlinear partial differential equations, and seems hard to find genuine 2-dimensional solutions. However, we have found interesting 1-dimensional solutions. In order to construct solutions, we make the ansatz,

$$\phi_3 = \psi \frac{i}{2} (\sigma_1 - i\sigma_2), \quad \phi_4 = \eta \frac{i}{2} (\sigma_1 + i\sigma_2), \quad \chi_1 = \psi i\sigma_3, \quad \chi_2 = -\eta i\sigma_3, \quad (5.18)$$

and set $\xi = 0$, $\partial_2 = 0$ and $A_1 = A_2 = 0$. The Spin(7)-instanton equations simplify to

$$\partial_1 \eta = -4\psi\eta \quad (5.19)$$

$$\partial_1 \psi = \frac{1}{R_1} \psi - \frac{1}{R_2} \eta + 2(\psi^2 - \eta^2) . \quad (5.20)$$

These flow equations have three fixed points P_i :

$$P_1 : (\psi, \eta) = (0, 0) , \quad P_2 : (\psi, \eta) = (-1/(2R_1), 0) , \quad P_3 : (\psi, \eta) = (0, -1/(2R_2)) . \quad (5.21)$$

The qualitative features of the solutions of (5.19), (5.20) depend on the value of the ratio $\rho = R_1/R_2$. For all ρ , there is a solution parametrised by a real constant x_0^1 ,

$$(\psi, \eta) = \left(1 - \exp \left(-\frac{1}{R_1} (x^1 - x_0^1) \right) \right)^{-1} \left(-\frac{1}{2R_1}, 0 \right) , \quad (5.22)$$

which asymptotes to P_1 at $x^1 = -\infty$ and P_2 at $x^1 = \infty$. This is in fact the well-known solution of the anti-self-dual Yang-Mills equations on $S^3 \times \mathbb{R}$, lifted to $S^3 \times S^3 \times S^1 \times \mathbb{R}$. When $\rho = \sqrt{3}$, there is also an analytic solution,

$$(\psi, \eta) = \left(-\frac{1}{2R_1}, 0 \right) + \frac{1}{2R_1} \left(1 + \exp \left(-\frac{2}{R_1} (x^1 - x_0^1) \right) \right)^{-1} \left(1, -\frac{1}{\sqrt{3}} \right) , \quad (5.23)$$

which asymptotes to P_2 at $x^1 = -\infty$ and P_3 at $x^1 = \infty$. For other values of ρ we have constructed solutions numerically, see figure 1. For $\rho < \sqrt{3}$ there exists a solution asymptoting to P_1 at $x^1 = -\infty$ and P_3 at $x^1 = \infty$, but for $\rho \geq \sqrt{3}$ this solution ceases to exist. Analysis of the critical points P_1, P_2, P_3 tells us that no other solutions asymptoting to the P_i at $x^1 = \pm\infty$ exist.

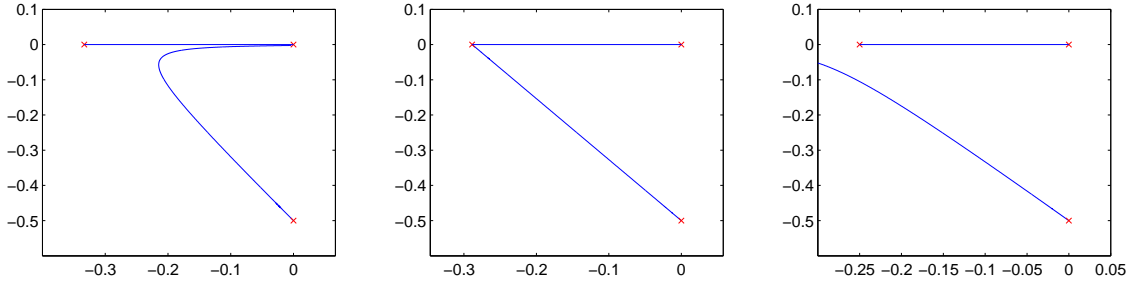


Figure 1: Solutions of the flow equation in the plane spanned by ψ (horizontal axis) and η (vertical axis) with $R_2 = 1$ and $R_1 = 1.5$ (left), $R_1 = \sqrt{3} \approx 1.73$ (centre), and $R_1 = 2$ (right)

As a point of interest, our ansatz (5.18) has implicitly imposed invariance of \mathcal{A} under the right action of SU(2). It is surprising that the equations are consistent with this ansatz, since the SU(4)-structure (5.10) is not invariant under the right action of SU(2). The explanation is that there is another SU(4)-structure which is invariant under the right action of SU(2):

$$\tilde{\Theta}^1 = \hat{e}^1 + i\hat{e}^1, \quad \tilde{\Theta}^2 = \hat{e}^2 - i\hat{e}^2, \quad \tilde{\Theta}^3 = \hat{e}^3 - i\hat{e}^3, \quad \tilde{\Theta}^4 = dx^1 + idx^2 . \quad (5.24)$$

We denote by ω and $\tilde{\omega}$ the (1,1)-forms determined by the frames $\Theta^\mu, \tilde{\Theta}^\mu$. Then

$$\tilde{\omega} = \text{Re}(\Theta^1 \wedge \Theta^2 + \Theta^3 \wedge \Theta^4) , \quad (5.25)$$

$$\omega = \text{Re}(\tilde{\Theta}^1 \wedge \tilde{\Theta}^2 - \tilde{\Theta}^3 \wedge \tilde{\Theta}^4) . \quad (5.26)$$

So $\tilde{\omega}$ solves the complex self-dual equation (3.9) for the frame Θ^μ with $\xi = 0$, and ω solves the complex self-dual equation for the frame $\tilde{\Theta}^\mu$ with $\xi = \pi$. Therefore the Spin(7)-instanton equations for the frame Θ^μ with $\xi = 0$ are equivalent to the Spin(7)-instanton equations for the frame $\tilde{\Theta}^\mu$ with $\xi = \pi$.

5.4 Invariant gauge fields on $S^3 \times S^1 \times T^4$

Now we turn our attention to $S^3 \times T^5$. Let e^1, e^2, e^3 be three left-invariant 1-forms on SU(2) as above, scaled so that $de^a = -(1/2R)\epsilon_{abc}e^b \wedge e^c$, and let e^4 be an invariant 1-form on U(1). We define an SU(4)-structure on $S^3 \times S^1 \times T^4$ by

$$\Theta^1 = e^1 + ie^2, \quad \Theta^2 = e^3 + ie^4, \quad \Theta^3 = dz^1, \quad \Theta^4 = dz^2, \quad (5.27)$$

with z^1, z^2 complex coordinates on T^4 . This structure is CYT, because the canonical torsion is

$$T = -\frac{1}{R} e^1 \wedge e^2 \wedge e^3 = -\frac{i}{4R} \Theta^1 \wedge \Theta^{\bar{1}} \wedge (\Theta^2 + \Theta^{\bar{2}}). \quad (5.28)$$

A gauge field on $T^4 \times S^3 \times S^1$ invariant under the left action of $SU(2) \times U(1)$ may be written using the ansatz (4.8), with A a gauge field on T^4 taking values in \mathfrak{k} and

$$\Phi = \phi_i \Theta^i + \phi_{\bar{i}} \Theta^{\bar{i}}, \quad i = 1, 2. \quad (5.29)$$

Here ϕ_i are two scalars on T^4 taking values in $\mathfrak{k}_{\mathbb{C}}$. Then the Spin(7)-instanton equations for \mathcal{F} on $S^3 \times S^1 \times T^4$ reduce to the equations,

$$e^{i\xi} D_{\bar{2}} \phi_{\bar{2}} = D_1 \phi_1, \quad (5.30)$$

$$-e^{i\xi} D_{\bar{2}} \phi_{\bar{1}} = D_1 \phi_2, \quad (5.31)$$

$$-e^{i\xi} F_{\bar{1}\bar{2}} = \frac{i}{2R} \phi_1 + [\phi_1, \phi_2], \quad (5.32)$$

$$F_{1\bar{1}} + F_{2\bar{2}} = -[\phi_1, \phi_{\bar{1}}] - [\phi_2, \phi_{\bar{2}}] + \frac{i}{2R} (\phi_2 + \phi_{\bar{2}}), \quad (5.33)$$

on T^4 (or one of its decompactifications $T^{4-p} \times \mathbb{R}^p$).

5.5 Kapustin-Witten equations

An alternative way to carry out the reduction from section 5.4 is to pick a different SU(4)-structure compatible with the same Spin(7)-structure. Consider again $S^3 \times S^1 \times T^4$ (or more generally, $S^3 \times S^1 \times T^{4-p} \times \mathbb{R}^p$). Let x^μ be real coordinates on T^4 , so that $z^1 = x^1 - ix^2$ and $z^2 = x^3 - ix^4$, and let

$$\tilde{\Theta}^\mu = dx^\mu + ie^\mu. \quad (5.34)$$

We denote by ω and $\tilde{\omega}$ the (1,1)-forms determined by the frames $\Theta^\mu, \tilde{\Theta}^\mu$. Then

$$\tilde{\omega} = -\text{Re}(\Theta^1 \wedge \Theta^3 + \Theta^{\bar{2}} \wedge \Theta^{\bar{4}}), \quad (5.35)$$

$$\omega = -\text{Re}(\tilde{\Theta}^1 \wedge \tilde{\Theta}^2 + \tilde{\Theta}^{\bar{3}} \wedge \tilde{\Theta}^{\bar{4}}). \quad (5.36)$$

So $\tilde{\omega}$ solves the complex self-dual equation (3.9) for the frame Θ^μ with $\xi = \pi$, and ω solves the complex self-dual equation for the frame $\tilde{\Theta}^\mu$ with $\xi = 0$. Therefore the Spin(7)-instanton equations for the frame $\tilde{\Theta}^\mu$ with $\xi = 0$ are equivalent to the Spin(7)-instanton equations for the frame Θ^μ with $\xi = \pi$. With respect to the frame $\tilde{\Theta}^\mu$, the canonical torsion is still a 3-form, but is neither nearly Kähler nor CYT:

$$T = -\frac{1}{R}e^1 \wedge e^2 \wedge e^3 = \frac{i}{8R}(\tilde{\Theta}^{\bar{1}} - \tilde{\Theta}^1) \wedge (\tilde{\Theta}^{\bar{2}} - \tilde{\Theta}^2) \wedge (\tilde{\Theta}^{\bar{3}} - \tilde{\Theta}^3) . \quad (5.37)$$

We make the same ansatz (4.8) for the gauge field, but parametrise Φ in terms of four \mathfrak{k} -valued scalars χ_μ rather than the two $\mathfrak{k}_\mathbb{C}$ -valued scalars ϕ_1, ϕ_2 :

$$\Phi = \chi_\mu e^\mu , \quad \mu = 1, \dots, 4 . \quad (5.38)$$

If we write $\chi = \chi_\mu dx^\mu$, the complex anti-self-dual equation (3.3) is reduced to the following equations in 4 dimensions:

$$\cos \frac{\xi}{2} \left(F - \chi \wedge \chi + \frac{1}{R} * (\chi \wedge dx^4) \right) - \sin \frac{\xi}{2} D\chi \quad \text{is anti-self-dual} , \quad (5.39)$$

$$\cos \frac{\xi}{2} D\chi + \sin \frac{\xi}{2} \left(F - \chi \wedge \chi + \frac{1}{R} * (\chi \wedge dx^4) \right) \quad \text{is self-dual} . \quad (5.40)$$

The second of the Spin(7)-instanton equations (3.4) becomes

$$D * \chi = 0 . \quad (5.41)$$

Equations (5.39)-(5.41) with $\xi = 0$ are equivalent to equations (5.30)-(5.33) with $\xi = \pi$.

The Hermitian-Yang-Mills equations are slightly stronger: they consist of (5.41), and the vanishing of the (2,0) part of \mathcal{F} :

$$\left(F - \chi \wedge \chi + \frac{1}{R} * (\chi \wedge dx^4) \right) = 0 , \quad (5.42)$$

$$D\chi = 0 . \quad (5.43)$$

Note that equations (5.39)-(5.43) on \mathbb{R}^4 or T^4 agree with the Kapustin-Witten equations (3.18)-(3.22) in the limit $R \rightarrow \infty$. So these equations are a simple, geometrically-motivated perturbation of Kapustin and Witten's – it would be interesting to see whether a topological field theory can be developed from them.

The Kapustin-Witten equations (3.18)-(3.20) only have non-trivial solutions in the cases $t = 0, \infty$. We have not found any solutions to our equations (5.39)-(5.41) on T^4 or $T^{4-p} \times \mathbb{R}^p$ (besides the usual Yang-Mills instantons); however, the Spin(7)-instanton constructed on $S^3 \times S^3 \times S^1 \times \mathbb{R}$ in section 5.3 corresponds to a solution of (5.39)-(5.41) on $S^3 \times \mathbb{R}$ invariant under rotations of S^3 .

6 Yang-Mills fields on the SU(2)-structure CYT spaces

In this section we discuss the Spin(7)-instanton equations on two CYT coset spaces whose natural structure group is SU(2).

6.1 Invariant gauge fields on $(S^5 \times S^1) \times T^2$

The coset space $SU(3)/SU(2)$ is S^5 as a manifold. A local basis of 1-forms e^a can be constructed from generators I_a as in section 4. We choose the quadratic form

$$q(X, Y) = -\frac{R^2}{4} \langle X, Y \rangle_{CK} = -\frac{3R^2}{2} \text{Tr}_3(XY) \quad (6.1)$$

on $\mathfrak{su}(3)$, with $\langle \cdot, \cdot \rangle_{CK}$ denoting the Cartan-Killing form. An orthonormal basis for \mathfrak{m} is

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{3}R} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \frac{1}{\sqrt{3}R} \begin{pmatrix} 0 & 0 & -1 \\ 0 & 0 & 0 \\ 1 & 0 & 0 \end{pmatrix}, \quad I_5 = \frac{1}{3R} \begin{pmatrix} 2i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & -i \end{pmatrix}, \\ I_2 &= \frac{1}{\sqrt{3}R} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_4 = \frac{1}{\sqrt{3}R} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}. \end{aligned} \quad (6.2)$$

The corresponding local frame e^a on S^5 is orthonormal with respect to the metric induced by q .

As explained in section 4, there is a natural $SU(2)$ -structure on $S^5 = SU(3)/SU(2)$. This $SU(2)$ -structure can be lifted to an $SU(4)$ -structure on $S^5 \times S^1 \times T^2$, defined by

$$\Theta^1 = e^1 + ie^2, \quad \Theta^2 = e^3 + ie^4, \quad \Theta^3 = e^5 + idt, \quad \Theta^4 = dz, \quad (6.3)$$

with t a coordinate on S^1 and z a complex coordinate on T^2 . The torsion (4.6) of this $SU(4)$ -structure is

$$T = \frac{1}{R} (e^1 \wedge e^2 + e^3 \wedge e^4) \wedge e^5 = \frac{i}{4R} (\Theta^1 \wedge \bar{\Theta}^1 + \Theta^2 \wedge \bar{\Theta}^2) \wedge (\Theta^3 + \bar{\Theta}^3). \quad (6.4)$$

Since T is the real part of a $(2,1)$ -form, the manifold is CYT. It is worth stressing that the metric on S^5 which makes $S^5 \times S^1$ CYT is not the usual round metric.

We make the standard ansatz (4.8) for an $SU(3)$ -invariant gauge field, with

$$\Phi = \phi_i \Theta^i + \bar{\phi}_{\bar{i}} \bar{\Theta}^{\bar{i}} + \chi e^5, \quad i = 1, 2, \quad (6.5)$$

where ϕ_i take values in $\mathfrak{k}_{\mathbb{C}}$ and χ take values in \mathfrak{k} , and both are subject to the constraint (4.7). The $\text{Spin}(7)$ -instanton equations on $S^5 \times S^1 \times T^2$ for the field strength (4.11) reduce to monopole-type equations,

$$iF_{zt} - D_z \chi = -2e^{i\xi} [\phi_{\bar{1}}, \phi_{\bar{2}}], \quad (6.6)$$

$$2D_z \phi_2 = e^{i\xi} \left(iD_t \phi_{\bar{1}} + [\chi, \phi_{\bar{1}}] - \frac{i}{2R} \phi_{\bar{1}} \right), \quad (6.7)$$

$$2D_z \phi_1 = -e^{i\xi} \left(iD_t \phi_{\bar{2}} + [\chi, \phi_{\bar{2}}] - \frac{i}{2R} \phi_{\bar{2}} \right), \quad (6.8)$$

$$2iF_{z\bar{z}} + D_t \chi - \frac{2}{R} \chi = -2i[\phi_i, \phi_{\bar{i}}], \quad (6.9)$$

on T^3 (or one of its decompactifications $\mathcal{T}^{3-p} \times \mathbb{R}^p$).

In principle one could choose any gauge group K , but to give a concrete example we make the choice $K = \text{SU}(3)$, and take the homomorphism from $\text{SU}(2)$ to $\text{SU}(3)$ to be the standard inclusion. The reduced gauge group is $\text{U}(1)$, the centraliser of $\text{SU}(2)$ in $\text{SU}(3)$, and we set $A = RaI_5$, with a a real 1-form. The constraint (4.7) is solved by

$$\phi_i = \phi RY_i, \quad \chi = \psi RI_5, \quad (6.10)$$

with ϕ, ψ two complex scalars and $Y_i := \frac{1}{2}(I_{2i-1} - iI_{2i})$. With these choices, the $\text{Spin}(7)$ -instanton equations reduce to

$$(\text{da})_{zt} + i\partial_z\psi = 0 \quad (6.11)$$

$$\partial_z\phi - ia_z\phi = 0 \quad (6.12)$$

$$\partial_t\phi - ia_t\phi + \left(\psi - \frac{1}{R}\right)\phi = 0 \quad (6.13)$$

$$2i(\text{da})_{z\bar{z}} + 2|\phi|^2 + \partial_t\psi - \frac{2}{R}\psi = 0. \quad (6.14)$$

These equations imply not just the $\text{Spin}(7)$ -instanton equations, but also the Hermitian-Yang-Mills equations. To find solutions, we assume t -invariance and write $\eta = \psi - ia_t$. The equations reduce further to

$$\partial_z\bar{\phi} - ia_z\bar{\phi} = 0, \quad (6.15)$$

$$\partial_z\bar{\eta} = 0, \quad (6.16)$$

$$\phi\left(\eta - \frac{1}{R}\right) = 0, \quad (6.17)$$

$$*\text{da} = 2|\phi|^2 - \frac{2}{R}\text{Re}(\eta). \quad (6.18)$$

These equations force either $\phi = 0$ or $\eta = 1/R$; the latter case is equivalent to the Bogmolny equation of the abelian Higgs model, which is known to admit many interesting solutions on both \mathbb{R}^2 and T^2 .

6.2 Invariant gauge fields on $S^5 \times S^3$

As above, we introduce a local frame of 1-forms on S^5 , now denoted \hat{e}^a with $a = 1, \dots, 5$. We denote by \check{e}^a , $a = 1, 2, 3$, the left-invariant 1-forms on $\text{SU}(2)$ satisfying $d\check{e}^a = (1/2\Lambda)\epsilon_{abc}\check{e}^b \wedge \check{e}^c$. An $\text{SU}(4)$ -structure on $S^5 \times S^3$ is given by,

$$\Theta^1 = \hat{e}^1 + i\hat{e}^2, \quad \Theta^2 = \hat{e}^3 + i\hat{e}^4, \quad \Theta^3 = \check{e}^1 + i\check{e}^2, \quad \Theta^4 = \hat{e}^5 + i\check{e}^3, \quad (6.19)$$

and this lifts the natural $\text{SU}(2)$ -structure described in section 4. The canonical torsion is the real part of a (2,1)-form, so the $\text{SU}(4)$ -structure is CYT:

$$T = \frac{1}{R}(\hat{e}^1 \wedge \hat{e}^2 + \hat{e}^3 \wedge \hat{e}^4) \wedge \hat{e}^5 + \frac{1}{\Lambda}\check{e}^1 \wedge \check{e}^2 \wedge \check{e}^3 \quad (6.20)$$

$$= \frac{i}{4R}(\Theta^1 \wedge \Theta^{\bar{1}} + \Theta^2 \wedge \Theta^{\bar{2}}) \wedge (\Theta^4 + \Theta^{\bar{4}}) + \frac{1}{4\Lambda}\Theta^3 \wedge \Theta^{\bar{3}} \wedge (\Theta^4 - \Theta^{\bar{4}}). \quad (6.21)$$

We make the usual ansatz (4.8) for the gauge field, with $A = 0$ and

$$\Phi = \phi_\mu \Theta^\mu + \phi_{\bar{\mu}} \Theta^{\bar{\mu}}, \quad \mu = 1, \dots, 4, \quad (6.22)$$

where ϕ_μ take values in $\mathfrak{k}_{\mathbb{C}}$ and are subject to the constraint (4.7). The Spin(7)-instanton equations are reduced to matrix model equations in $d = 0$ dimensions:

$$e^{i\xi}[\phi_{\bar{1}}, \phi_{\bar{2}}] = [\phi_4, \phi_3] + \frac{1}{2\Lambda} \phi_3, \quad (6.23)$$

$$e^{i\xi}[\phi_{\bar{2}}, \phi_{\bar{3}}] = [\phi_4, \phi_1] + \frac{i}{2R} \phi_1, \quad (6.24)$$

$$e^{i\xi}[\phi_{\bar{3}}, \phi_{\bar{1}}] = [\phi_4, \phi_2] + \frac{i}{2R} \phi_3, \quad (6.25)$$

$$2i \sum_{\mu=1}^4 [\phi_\mu, \phi_{\bar{\mu}}] = \left(\frac{2}{R} - \frac{i}{\Lambda}\right) \phi_4 + \left(\frac{2}{R} + \frac{i}{\Lambda}\right) \phi_{\bar{4}}. \quad (6.26)$$

We choose gauge group $K = \text{SU}(3)$, with $\text{SU}(2) \subset \text{SU}(3)$ embedded in the obvious way. The constraint (4.7) is solved by

$$\phi_1 = \phi Y_1, \quad \phi_2 = \phi Y_2, \quad \phi_3 = \psi I_7, \quad \phi_4 = \chi I_7, \quad (6.27)$$

for complex ϕ, χ, ψ . The Spin(7)-instanton equations become $\psi = 0$ and

$$\phi\left(\chi - \frac{1}{2}\right) = 0 \quad (6.28)$$

$$|\phi|^2 = \text{Re}\left((2 - i\frac{R}{\Lambda})\chi\right). \quad (6.29)$$

Once again, these equations imply the Hermitian-Yang-Mills equations as well as the Spin(7)-instanton equations. They have two types of solution: either $\phi = 0$ and $\chi = \lambda(R/\Lambda - 2i)$ for some $\lambda \in \mathbb{R}$; or $\chi = \frac{1}{2}$ and $|\phi|^2 = 1$ (the canonical connection $\phi = 0, \chi = 0$ is included in the former).

7 Yang-Mills fields on SU(3)-structure CYT spaces

In this section, we study the Spin(7)-instanton equations on the CYT manifold $S^7 \times S^1$. As a manifold, S^7 is the coset space $\text{SU}(4)/\text{SU}(3)$, so the natural structure group is $\text{SU}(3)$. Following the formalism described in section 4, we choose a quadratic form on $\mathfrak{su}(4)$:

$$q(X, Y) = -\frac{R^2}{6} \langle X, Y \rangle_{CK} = -\frac{4R^2}{3} \text{Tr}_4(XY). \quad (7.1)$$

An orthonormal basis for \mathfrak{m} is

$$\begin{aligned} I_1 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & -1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_3 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_5 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}, \\ I_2 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_4 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & 0 & i & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}, & I_6 &= \frac{\sqrt{6}}{4R} \begin{pmatrix} 0 & 0 & 0 & i \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ i & 0 & 0 & 0 \end{pmatrix}, \\ I_7 &= \frac{1}{4R} \begin{pmatrix} 3i & 0 & 0 & 0 \\ 0 & -i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix}. \end{aligned} \quad (7.2)$$

This induces a local frame e^1, \dots, e^7 of 1-forms. An $SU(4)$ -structure on $S^7 \times S^1$ which lifts the natural $SU(3)$ -structure is induced by

$$\Theta^\alpha = e^{2\alpha-1} + ie^{2\alpha}, \quad \Theta^4 = e^7 + \text{idt}, \quad (7.3)$$

with $\alpha = 1, 2, 3$ and t a coordinate on S^1 . The canonical torsion is the real part of a $(2,1)$ -form, so the $SU(4)$ -structure is CYT:

$$T = \frac{1}{R}(e^1 \wedge e^2 + e^3 \wedge e^4 + e^5 \wedge e^6) \wedge e^7 = \frac{i}{4R} \Theta^\alpha \wedge \Theta^{\bar{\alpha}} \wedge (\Theta^4 + \Theta^{\bar{4}}). \quad (7.4)$$

We make the usual ansatz for an $SU(4)$ -invariant gauge field (4.8), with

$$\Phi = \phi_\alpha \Theta^\alpha + \phi_{\bar{\alpha}} \Theta^{\bar{\alpha}} + \chi e^7, \quad \alpha = 1, 2, 3, \quad (7.5)$$

and the gauge chosen so that $A = 0$. The scalar χ takes values in the Lie algebra \mathfrak{k} of K and the scalars ϕ_α take values in $\mathfrak{k}_{\mathbb{C}}$; together they must satisfy the constraint (4.7). The $\text{Spin}(7)$ -instanton equations are then reduced to the equations,

$$\frac{i}{2} \left(\dot{\phi}_\alpha - \frac{1}{R} \phi_\alpha + i[\chi, \phi_\alpha] \right) = -\frac{1}{2} e^{i\xi} \epsilon_{\alpha\beta\gamma} [\phi_{\bar{\beta}}, \phi_{\bar{\gamma}}], \quad (7.6)$$

$$\dot{\chi} - \frac{3}{R} \chi = -2i[\phi_\alpha, \phi_{\bar{\alpha}}], \quad (7.7)$$

on S^1 (or its decompactification \mathbb{R}). We choose gauge group $SU(4)$ with $SU(3)$ embedded in the obvious way, so that the constraint (4.7) is solved by

$$\phi_\alpha = \phi R Y_\alpha, \quad \chi = \psi R I_7, \quad Y_\alpha = \frac{1}{2}(I_{2\alpha-1} - iI_{2\alpha}). \quad (7.8)$$

Then the $\text{Spin}(7)$ -instanton equations reduce to

$$\dot{\phi} = \phi \left(\frac{1}{R} - \psi \right) \quad (7.9)$$

$$\dot{\psi} = 3 \left(\frac{1}{R} \psi - |\phi|^2 \right). \quad (7.10)$$

These equations imply the Hermitian-Yang-Mills equations as well as the $\text{Spin}(7)$ -instanton equations. They are gradient flow equations for a function W :

$$\dot{\phi} = \frac{\partial W}{\partial \bar{\phi}} \quad (7.11)$$

$$\dot{\psi} = 3 \frac{\partial W}{\partial \psi} \quad (7.12)$$

$$W = \frac{1}{R} |\phi|^2 + \frac{1}{2R} \psi^2 - \psi |\phi|^2. \quad (7.13)$$

Since the value of W decreases along the flow, the only solutions with $t \in S^1$ are t -independent: either $\psi = 0, \phi = 0$ (the canonical connection) or $\psi = 1/R, |\phi| = 1/R$. It is also interesting to look for solutions with $t \in \mathbb{R}$ which interpolate between the critical points of W . The critical point $\psi = 0, \phi = 0$ is a local minimum for W , and the critical point $\psi = 1/R, |\phi| = 1/R$ is a saddle point; it follows that there is a unique (up to translation) solution which tends to $\psi = 1/R, |\phi| = 1/R$ as $t \rightarrow -\infty$ and to $\psi = 0, \phi = 0$ as $t \rightarrow \infty$.

8 Yang-Mills fields on nearly Kähler manifolds and vortex equations

8.1 Reduced Spin(7)-instanton equations

There are only four known examples of nearly Kähler 6-manifolds, and they are all cosets [21]:

$$\begin{aligned} \mathrm{SU}(3)/\mathrm{U}(1) \times \mathrm{U}(1) , \quad & \mathrm{Sp}(2)/\mathrm{Sp}(1) \times \mathrm{U}(1) , \\ G_2/\mathrm{SU}(3) = S^6 , \quad & \mathrm{SU}(2)^3/\mathrm{SU}(2) = S^3 \times S^3 . \end{aligned} \quad (8.1)$$

We will consider the dimensional reduction of the Spin(7)-instanton equations over these spaces.

The $\mathrm{SU}(3)$ -structure on the coset space is determined in an elegant way from the Lie algebra \mathfrak{g} . For example, the metric g is determined by the Cartan-Killing form, and the complex structure J is related to the 3-symmetry, a linear map $S : \mathfrak{m} \rightarrow \mathfrak{m}$ satisfying $S^3 = 1$. Details of this construction can be found in [13, 21].

Here we will specify the nearly Kähler $\mathrm{SU}(3)$ -structures on the coset spaces more directly. We choose the quadratic form,

$$q(X, Y) = -\frac{R^2}{12} \langle X, Y \rangle_{CK} , \quad (8.2)$$

on all four coset spaces. For each coset space, we will write down an orthonormal basis I_a for \mathfrak{m} . Then left-invariant 1-forms e^a are induced on the coset spaces as described in section 4, and we can define

$$\Theta^\alpha = e^{2\alpha-1} + i e^{2\alpha} , \quad \alpha = 1, 2, 3 . \quad (8.3)$$

The $\mathrm{SU}(3)$ -structure is then determined by J, ω, Ω written in the standard form (2.2). An $\mathrm{SU}(4)$ structure on $G/H \times T^2$ is defined by setting $\Theta^4 = dz$, with z a complex coordinate on T^2 . It will prove convenient to define $Y_\alpha = \frac{1}{2}(I_{2\alpha-1} - iI_{2\alpha})$. In section 8.5 we will use different conventions.

The four nearly Kähler coset spaces are algebraically very similar; for example, the torsion tensor (4.6) is

$$T = -\frac{1}{6} f_{abc} e^a \wedge e^b \wedge e^c = \frac{1}{R} \mathrm{Re} \Omega \quad (8.4)$$

This means that the reduction of Spin(7)-instanton equations depends on the choice of nearly Kähler coset space only through the constraint (4.7).

We will make the usual ansatz for a G -invariant gauge field (4.8), with $\Phi = \phi_\alpha \Theta^\alpha + \phi_{\bar{\alpha}} \Theta^{\bar{\alpha}}$ and ϕ_α scalars on T^2 taking values in the complexification of the Lie algebra of K . The Spin(7)-instanton equations for \mathcal{F} are then reduced to the equations,

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{\alpha}} = \frac{1}{2} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} [\phi_{\bar{\beta}}, \phi_{\bar{\gamma}}] + \frac{1}{R} \phi_{\bar{\alpha}} , \quad (8.5)$$

$$*F = 2i[\phi_\alpha, \phi_{\bar{\alpha}}] , \quad (8.6)$$

on T^2 . The field strength F^0 of the canonical connection solves the Spin(7)-instanton equations on a nearly Kähler coset space, so it does not appear in these equations. Thus the only differences between these equations and (3.25), (3.26) come from the torsion term in (4.11) and the constraint (4.7) on the fields ϕ_α .

The real metric q on \mathfrak{g} can be extended to define a Hermitian metric on $\mathfrak{g}_{\mathbb{C}}$. Then, similar to equation (3.25), equation (8.5) is the Hermitian flow equation for the function

$$W(\phi_1, \phi_2, \phi_3) = q(\phi_1, [\phi_2, \phi_3]) + q(\phi_{\bar{1}}, [\phi_{\bar{2}}, \phi_{\bar{3}}]) + \frac{1}{R} q(\phi_\alpha, \phi_{\bar{\alpha}}) . \quad (8.7)$$

Now we will give examples of the Spin(7)-instanton equations on the SU(4)-structure manifold $T^2 \times G/H$ for each of the four nearly Kähler cosets G/H .

8.2 Invariant gauge fields on $T^2 \times G_2/\text{SU}(3)$

This case was discussed in [13]. An obvious choice of gauge group is G_2 , and a bundle over S^6 admitting an action of G_2 is determined by the obvious embedding of $\text{SU}(3)$ in G_2 . Then the reduced gauge group is trivial, and the solution of the constraint (4.7) is parametrised by a complex scalar ϕ . The Spin(7)-instanton equations (8.5), (8.6) reduce to

$$e^{i\xi} \partial_{\bar{z}} \bar{\phi} = \frac{1}{R} (\bar{\phi} - \phi^2) . \quad (8.8)$$

8.3 Invariant gauge fields on $T^2 \times \text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$

We choose the following normalised basis for \mathfrak{m}

$$\begin{aligned} I_1 &= \frac{1}{\sqrt{2}R} \begin{pmatrix} 0 & -1_2 \\ 1_2 & 0 \end{pmatrix}, & I_3 &= \frac{1}{\sqrt{2}R} \begin{pmatrix} 0 & -i\sigma^1 \\ -i\sigma^1 & 0 \end{pmatrix}, & I_5 &= \frac{1}{R} \begin{pmatrix} 0 & 0 \\ 0 & i\sigma^1 \end{pmatrix}, \\ I_2 &= \frac{1}{\sqrt{2}R} \begin{pmatrix} 0 & -i\sigma^3 \\ -i\sigma^3 & 0 \end{pmatrix}, & I_4 &= \frac{1}{\sqrt{2}R} \begin{pmatrix} 0 & -i\sigma^2 \\ -i\sigma^2 & 0 \end{pmatrix}, & I_6 &= \frac{1}{R} \begin{pmatrix} 0 & 0 \\ 0 & -i\sigma^2 \end{pmatrix}. \end{aligned} \quad (8.9)$$

The nearly Kähler SU(3)-structure on $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$ is written in the standard form (8.3) in terms of the 1-forms e^a induced by the I_a .

We choose gauge group $K = \text{Sp}(2)$ and take the homomorphism from $\text{Sp}(1) \times \text{U}(1)$ to be the inclusion. The reduced gauge group is the centraliser $\text{U}(1)$ of $\text{Sp}(1) \times \text{U}(1)$, and the gauge field on T^2 will be written

$$A = a \begin{pmatrix} 0 & 0 \\ 0 & -i\sigma^3 \end{pmatrix} \quad (8.10)$$

with a a real 1-form. The solution of the condition (4.7) is

$$\phi_1 = \phi Y_1, \quad \phi_2 = \phi Y_2, \quad \phi_3 = \chi Y_3, \quad (8.11)$$

with ϕ, χ two complex scalars.

There is actually a 1-parameter family of Spin(2)-invariant SU(3)-structures on the space $\text{Sp}(2)/\text{Sp}(1) \times \text{U}(1)$, which includes the nearly Kähler structure. With respect to the fixed frame Θ^α , this family is written

$$\omega = \frac{i}{2} (\Lambda^2 \Theta^1 \wedge \bar{\Theta}^1 + \Lambda^2 \Theta^2 \wedge \bar{\Theta}^2 + \Lambda^{-2} \Theta^3 \wedge \bar{\Theta}^3), \quad \Omega = \Lambda \Theta^1 \wedge \Theta^2 \wedge \Theta^3, \quad (8.12)$$

with Λ a positive real parameter and g fixed by J, ω . The 2-form ω is a genuine Sp(2)-invariant form on the coset, because it is induced from an $\text{Sp}(1) \times \text{U}(1)$ -invariant form on \mathfrak{m} .

When $\Lambda = 1$ the SU(3)-structure is nearly Kähler, and the reduction of the Spin(7)-instanton equations is obtained by substituting for ϕ_α and A in (8.5), (8.6). For $\Lambda \neq 1$ the reduction of

the Spin(7)-instanton equations must be calculated directly from the formula (4.11) for the field strength. The result is:

$$(\partial_{\bar{z}} - ia_{\bar{z}})\bar{\phi} = \frac{\Lambda}{R}(\bar{\phi} - \phi\chi) , \quad (8.13)$$

$$(\partial_{\bar{z}} + 2ia_{\bar{z}})\bar{\chi} = \frac{1}{R\Lambda^3}(\bar{\chi} - \phi^2) , \quad (8.14)$$

$$*da = \frac{2}{R^2} \left(\frac{1}{\Lambda^2} - \Lambda^2 - \frac{|\phi|^2}{\Lambda^2} + \Lambda^2|\chi|^2 \right) . \quad (8.15)$$

8.4 Invariant gauge fields on $T^2 \times \text{SU}(3)/\text{U}(1) \times \text{U}(1)$

The normalised basis for \mathfrak{m} which yields the nearly Kähler structure in standard form (8.3) is

$$\begin{aligned} I_1 &= \frac{1}{R} \begin{pmatrix} 0 & -1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_3 = \frac{1}{R} \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ -1 & 0 & 0 \end{pmatrix}, \quad I_5 = \frac{1}{R} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \\ I_2 &= \frac{1}{R} \begin{pmatrix} 0 & i & 0 \\ i & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}, \quad I_4 = \frac{1}{R} \begin{pmatrix} 0 & 0 & i \\ 0 & 0 & 0 \\ i & 0 & 0 \end{pmatrix}, \quad I_6 = \frac{1}{R} \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & i \\ 0 & i & 0 \end{pmatrix}. \end{aligned} \quad (8.16)$$

We choose gauge group $K = \text{SU}(3)$, with $\text{U}(1) \times \text{U}(1)$ embedded using the inclusion. The condition (4.7) is solved by

$$\phi_1 = \chi_1 Y_1, \quad \phi_2 = \chi_2 Y_2, \quad \phi_3 = \chi_3 Y_3, \quad (8.17)$$

with χ_α complex scalars. The reduced gauge group is $\text{U}(1) \times \text{U}(1)$, and we set

$$A = a \begin{pmatrix} i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 0 \end{pmatrix} + \frac{b}{3} \begin{pmatrix} -i & 0 & 0 \\ 0 & -i & 0 \\ 0 & 0 & 2i \end{pmatrix} \quad (8.18)$$

for real 1-forms a, b .

Once again, we may consider a family of $\text{SU}(3)$ -structures parametrised by $\Lambda \in \mathbb{R}$, for which the nearly Kähler case appears when $\Lambda = 1$:

$$\omega = \frac{i}{2} (\Lambda^{-2} \Theta^1 \wedge \bar{\Theta}^1 + \Lambda^2 \Theta^2 \wedge \bar{\Theta}^2 + \Lambda^2 \Theta^3 \wedge \bar{\Theta}^3), \quad \Omega = \Lambda \Theta^1 \wedge \Theta^2 \wedge \Theta^3. \quad (8.19)$$

The reduction of the Spin(7)-instanton equations is obtained by substituting for ϕ_α and A in the formula (4.11) for the field strength, or in the case $\Lambda = 1$ simply by substituting into (8.5), (8.6).

The result is:

$$e^{i\xi}(\partial_{\bar{z}} + 2ia_{\bar{z}})\bar{\chi}_1 = \frac{1}{R\Lambda^3}(\bar{\chi}_1 - \chi_2\chi_3) , \quad (8.20)$$

$$e^{i\xi}(\partial_{\bar{z}} - ia_{\bar{z}} + ib_{\bar{z}})\bar{\chi}_2 = \frac{\Lambda}{R}(\bar{\chi}_2 - \chi_1\chi_3) , \quad (8.21)$$

$$e^{i\xi}(\partial_{\bar{z}} - ia_{\bar{z}} - ib_{\bar{z}})\bar{\chi}_3 = \frac{\Lambda}{R}(\bar{\chi}_3 - \chi_1\chi_2) , \quad (8.22)$$

$$*da = \frac{2}{R^2} \left(\frac{1}{\Lambda^2} - \Lambda^2 - \frac{|\chi_2|^2 + |\chi_3|^2}{2\Lambda^2} + \Lambda^2|\chi_1|^2 \right) , \quad (8.23)$$

$$*db = \frac{3}{R^2\Lambda^2}(|\chi_2|^2 - |\chi_3|^2) . \quad (8.24)$$

8.5 Invariant gauge fields on $T^2 \times \text{SU}(2)^3/\text{SU}(2)$

The Spin(7)-instanton equations can be reduced over $\text{SU}(3)^3/\text{SU}(2)$ using the formalism outlined in subsection 8.1, just as they can over the other three nearly Kähler coset spaces. We will not do so here; instead, we reduce the equations using a different scheme which allows us to be more flexible about what symmetries are imposed.

We begin our discussion by observing that the coset space $\text{SU}(3)^3/\text{SU}(2)$ is diffeomorphic to $\text{SU}(2) \times \text{SU}(2) = S^3 \times S^3$ as a manifold. The diffeomorphism is

$$(g_1, g_2, g_3)\text{SU}(2) \mapsto (g_1g_3^{-1}, g_2g_3^{-1}) . \quad (8.25)$$

The left action of $\text{SU}(2) \times \text{SU}(2) \times 1$ on the coset space coincides with the left action of $\text{SU}(2) \times \text{SU}(2)$ on itself, while the left action of $1 \times 1 \times \text{SU}(2)$ on the coset space corresponds to a right action of a diagonal $\text{SU}(2)$ on $\text{SU}(2) \times \text{SU}(2)$. We denote this latter action $\text{SU}(2)_R$. The basic idea is that one can impose symmetry under $\text{SU}(2)^2$ and $\text{SU}(2)_R$ separately.

Let $\hat{e}^\alpha, \check{e}^\alpha$ denote left-invariant 1-forms on two copies of $S^3 = \text{SU}(2)$, normalised so that

$$d\hat{e}^\alpha = -\frac{1}{2R}\epsilon_{\alpha\beta\gamma}\hat{e}^\beta \wedge \hat{e}^\gamma , \quad (8.26)$$

and similar for \check{e}^α . Both \hat{e}^α and \check{e}^α transform in the irreducible 3-dimensional representation of $\text{SU}(2)_R$. The nearly Kähler structure is determined by setting

$$\Theta^\alpha = \frac{1}{3}(\tau\hat{e}^\alpha + \bar{\tau}\check{e}^\alpha) , \quad \tau = -\frac{1}{2} + i\frac{\sqrt{3}}{2} . \quad (8.27)$$

Note that the corresponding metric is not the usual product metric on $S^3 \times S^3$. The forms Θ^α satisfy

$$d\Theta^\alpha = -\frac{1}{2R}(2\epsilon^{\alpha\bar{\beta}\bar{\gamma}}\Theta^\beta \wedge \Theta^\gamma - \epsilon^{\alpha\beta\gamma}\Theta^{\bar{\beta}} \wedge \Theta^{\bar{\gamma}} + 2\epsilon^{\alpha\beta\bar{\gamma}}\Theta^{\bar{\beta}} \wedge \Theta^\gamma) . \quad (8.28)$$

The (1,0)-forms Θ^α define an $\text{SU}(3)$ -structure written in the standard form (2.2) on $S^3 \times S^3$; the easiest way to see that it is nearly Kähler is to compute,

$$d\omega = \frac{3}{R}\text{Im}\Omega , \quad d\Omega = \frac{2}{R}\omega \wedge \omega . \quad (8.29)$$

Let A denote a gauge field on \mathbb{R}^2 with gauge group K and let ϕ_α denote three adjoint scalars. Then the following ansatz describes the most general possible $SU(2)^2$ -invariant gauge field on $\mathbb{R}^2 \times S^3 \times S^3$:

$$\mathcal{A} = A + \phi_\alpha \Theta^\alpha + \phi_{\bar{\alpha}} \Theta^{\bar{\alpha}} . \quad (8.30)$$

The Spin(7)-instanton equations for \mathcal{A} are reduced to

$$e^{i\xi} D_{\bar{z}} \phi_{\bar{\alpha}} = \frac{1}{2} \epsilon_{\bar{\alpha}\bar{\beta}\bar{\gamma}} [\phi_{\beta}, \phi_{\gamma}] + \frac{1}{R} (\phi_{\bar{\alpha}} - 2\phi_{\alpha}) , \quad (8.31)$$

$$*F = 2i[\phi_{\alpha}, \phi_{\bar{\alpha}}] . \quad (8.32)$$

Let us emphasise again that we have imposed only $SU(2)^2$ symmetry, whereas the method described in subsection 8.1 imposes $SU(2)^3$ symmetry. The additional $SU(2)_R$ symmetry can be imposed within the present method, if desired. For example, with gauge group $K = SU(2)$, the ansatz,

$$\phi_{\alpha} = \frac{i}{2R} (\phi - 1) \sigma_{\alpha} , \quad A = 0 , \quad (8.33)$$

with ϕ a complex scalar, is invariant under a combination of $SU(2)_R$ rotations and gauge transformations. Then the Spin(7)-instanton equations become

$$e^{i\xi} \partial_{\bar{z}} \bar{\phi} = \frac{1}{R} (\bar{\phi} - \phi^2) . \quad (8.34)$$

This is the equation that we would have obtained if we had followed the method of subsection 8.1.

8.6 Vortices and invariant instantons

We have already emphasised that the four nearly Kähler coset spaces are algebraically very similar. One consequence is that the reduced Spin(7)-instanton equations written down in the previous subsections are all related to each other. Clearly, the equations (8.8) and (8.34) arising in the cases $G_2/SU(3)$ and $SU(2)^3/SU(2)$ coincide. These in turn are obtained from equations (8.13)-(8.15) arising in the case $Sp(2)/Sp(1) \times U(1)$, on setting $\phi = \chi, a = 0$ and $\Lambda = 1$. Finally, the equations (8.20)-(8.24) arising in the case $SU(3)/U(1)^2$ specialise to (8.13)-(8.15) on setting $\chi_1 = \chi, \chi_2 = \chi_3 = \phi, b = 0$. As a result, solving any one of the above equations may lead to solutions of some of the others.

The Hermitian flow structure can help in the search for solutions. We start by discussing equation (8.8), which is a Hermitian flow for

$$h = \frac{3}{2} d\phi d\bar{\phi} , \quad W = \frac{1}{2R} (3|\phi|^2 - \phi^3 - \bar{\phi}^3) . \quad (8.35)$$

By making a rotation of $z = x + iy$ we can fix $\xi = 0$; then by imposing $\partial_y = 0$ or $\partial_x = 0$ the Hermitian flow reduces to a gradient flow or a Hamiltonian flow respectively. Solutions of these equations on \mathbb{R} were discussed in [13]; in the present context these lift to solutions of the Spin(7)-instanton equations on $M^6 \times \mathbb{R} \times S^1$, with M^6 any nearly Kähler coset space. Solutions of the Hamiltonian flow also exist on S^1 : these move around the closed trajectories $W = C$ for $C < 1/2R$. In the present context these give us Spin(7)-instantons on $M^6 \times T^2$.

Equations (8.13)-(8.15) with $\Lambda = 1$ are also a Hermitian flow, this time for

$$h = \frac{1}{2}(2d\phi d\bar{\phi} + d\chi d\bar{\chi}) , \quad W = \frac{1}{2R}(2|\phi|^2 + |\chi|^2 - \phi^2\chi - \bar{\phi}^2\bar{\chi}) . \quad (8.36)$$

The Hamiltonian flow preserves not only W , but also $U = |\phi|^2 - |\chi|^2$. This is because U is the moment map generating the $U(1)$ gauge symmetry of W . So the Hamiltonian flow is integrable, a fact which should help the search for solutions. Similar comments apply to equations (8.20)-(8.24).

Equations (8.13)-(8.15) also possess solutions with $a \neq 0$. Choose a gauge so that $\phi = e^f$ for some real function f , and suppose that $\chi = \bar{\phi}^2 = e^{2f}$. Then equation (8.14) implies that $a_{\bar{z}} = i\partial_{\bar{z}}f$. Equations (8.13) and (8.15) become

$$2\partial_{\bar{z}}f = \frac{1}{R}(1 - e^{2f}) , \quad (8.37)$$

$$4\partial_z\partial_{\bar{z}}f = \frac{2}{R^2}e^{2f}(e^{2f} - 1) . \quad (8.38)$$

Surprisingly the second of these equations is implied by the first, so we need only solve the first. A solution is

$$f = -\frac{1}{2}\ln(1 + e^{-2x/R}) . \quad (8.39)$$

This gives an $\text{Spin}(7)$ -instanton on $M^6 \times \mathbb{R} \times S^1$, with $M^6 = \text{Sp}(2)/\text{Sp}(1) \times U(1)$ or $\text{SU}(3)/U(1)^2$.

All of the solutions presented above are essentially 1-dimensional; we have not searched for genuine 2-dimensional solutions with $\Lambda = 1$. However, studying the vacua of these equations (i.e. the z -independent solutions) can give hints as to what 2-dimensional solutions might exist. First, equation (8.8) has a \mathbb{Z}_3 -symmetry, and its four vacua are $\phi = 0$ and $\phi = \text{a cube root of unity}$. It is quite similar to an equation studied in [28], except that the function $W = 3|\phi|^2 - \phi^3 - \bar{\phi}^3$ playing the role of superpotential is not holomorphic. Equation (8.8) might admit domain wall junction solutions, similar to those found in [28].

Second, the non-zero vacua of (8.13)-(8.15) or (8.20)-(8.24) form topologically non-trivial manifolds. The vacuum manifolds are $|\phi|^2 = 1$, $\chi = \bar{\phi}^2$ and $|\chi_2|^2 = |\chi_3|^2 = 1$, $\chi_1 = \bar{\chi}_2\bar{\chi}_3$, which are topologically a circle and 2-torus, respectively. Thus field configurations on \mathbb{R}^2 which tend to the vacuum at infinity possess a topological charge, the winding number of the field at infinity. It would be interesting to see whether topologically non-trivial solutions to these equations exist.

We have been able to find 2-dimensional solutions in the case where $\Lambda > 1$. Setting $\phi = 0$ in (8.13)-(8.15) gives

$$\left(\partial_{\bar{z}} - 2ia_{\bar{z}} - \frac{1}{\Lambda^3 R}\right)\bar{\chi} = 0 , \quad (8.40)$$

$$*da = \frac{2\Lambda^2}{R^2}\left(|\chi|^2 - 1 + \frac{1}{\Lambda^4}\right) . \quad (8.41)$$

For $\Lambda > 1$ these are mathematically equivalent to the BPS abelian Higgs vortex equations, with gauge field $\tilde{a} = a - (1/\Lambda^3 R)dy$. Thus solutions of the abelian Higgs vortex equations on Σ^2 yield $\text{Spin}(7)$ -instantons on $M^6 \times \Sigma^2$, with $M^6 = \text{Sp}(2)/\text{Sp}(1) \times U(1)$ or $\text{SU}(3)/U(1)^2$ and $\Sigma^2 = \mathbb{R}^2$, $\mathbb{R} \times S^1$ or T^2 .

9 Conclusions

We have studied the Spin(7)-instanton equations on 8-manifolds of the form $M^d \times G/H$, with $M^d = T^d$ or one of its decompactifications $T^{d-p} \times \mathbb{R}^p$. By imposing G -invariance, the Spin(7)-instanton equations reduce to equations on M^d . Where possible, we have constructed solutions, although our search was not exhaustive and we anticipate that further solutions could be found with more work. One could also take M^d to be any manifold admitting an $SU(n)$ -structure for $n \leq 2d$; our calculations should go through with only minor changes in this more general situation.

Since all of the compact spaces we considered were elliptic fibrations, Spin(7)-instantons on them are potentially relevant in F-theory compactifications, although compactification of F-theory on non-Calabi-Yau manifolds is far from being understood. Nevertheless, we hope that our examples have served to illustrate that there is something to be said about generalised instantons, and torsionful Yang-Mills theory, on manifolds with non-vanishing intrinsic torsion. Of course, the methods that we have used could easily be adapted to studying the G_2 -instanton and Hermitian-Yang-Mills equations in seven and six dimensions, which can be obtained from the Spin(7)-instanton equations by dimensional reduction. These equations are relevant to M- and string theory compactifications.

A natural direction for further study would be to solve the supersymmetry constraint equations of heterotic supergravity by using solutions to all these instanton equations. It is also of interest to extend the techniques of the equivariant dimensional reduction for Kähler coset spaces [29] to higher rank bundles over homogeneous manifolds with $SU(4)$ -structure.

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